DEFINITE INTEGRATION & AREA

DEFINITE INTEGRATION

1. DEFINITION

Let F (x) be any antiderivative of f(x), then for any two values of the independent variable x, say a and b, the difference F (b) – F (a) is called the definite integral of f(x) from a to b and

is denoted by $\int_a^b f(x)\,dx$. Thus $\int_a^b f(x)\,dx = F(b) - F(a)$,

The numbers a and b are called the limits of integration; a is the lower limit and b is the upper limit. Usually F (b) – F (a) is abbreviated by writing $F(x)\big|_a^b$.

2. PROPERTIES OF DEFINITE INTEGRALS

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)$

2. $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where c may or may not lie between a and b.

4. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

5. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Note...

1. $\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \frac{a}{2}$

2. $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$

6. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

$$= \begin{cases} 0 & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \end{cases}$$

7. $\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(-x) = f(x) \text{ i.e. } f(x) \text{ is even} \\ 0 & \text{if } f(-x) = -f(x) \text{ i.e. } f(x) \text{ is odd} \end{cases}$

8. If f (x) is a periodic function of period 'a', i.e. f(a + x) = f(x), then

(a) $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$

(b) $\int_a^{na} f(x)\,dx = (n-1)\int_0^a f(x)\,dx$

(c) $\int_{na}^{b+na} f(x)\,dx = \int_0^b f(x)\,dx$, where $b \in R$

(d) $\int_b^{b+a} f(x)\,dx$ independent of b.

(e) $\int_b^{b+na} f(x)\,dx = n\int_0^a f(x)\,dx$, where $n \in I$

9. If $f(x) \geq 0$ on the interval [a, b], then $\int_a^b f(x)\,dx \geq 0$.

10. If $f(x) \leq g(x)$ on the interval [a, b], then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

11. $$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

12. If $f(x)$ is continuous on [a, b], m is the least and M is the greatest value of $f(x)$ on [a, b], then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

13. For any two functions $f(x)$ and $g(x)$, integrable on the interval [a, b], the Schwarz – Bunyakovsky inequality holds

$$\left|\int_a^b f(x).g(x)\,dx\right| \leq \sqrt{\int_a^b f^2(x)\,dx.\int_a^b g^2(x)\,dx}$$

14. If a function $f(x)$ is continuous on the interval [a, b], then there exists a point $c \in (a, b)$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a), \text{ where } a < c < b$$

3. DIFFERENTIATION UNDER INTEGRAL SIGN

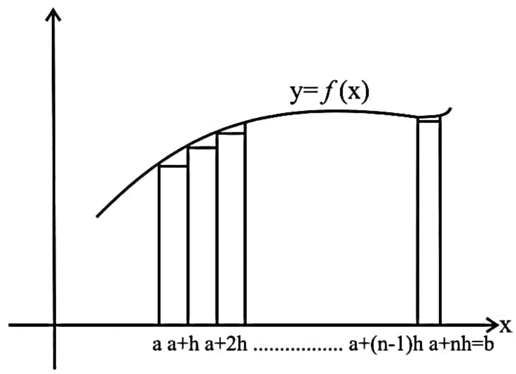
NEWTON LEIBNITZ'S THEOREM :

If f is continuous on [a, b] and g(x) & h(x) are differentiable functions of x whose values lie in [a, b], then

$$\frac{d}{dx}\left[\int_{g(x)}^{h(x)} f(t)\,dt\right] = \frac{d}{dx}[h(x)].f[h(x)] - \frac{d}{dx}[g(x)].f[g(x)]$$

4. DEFINITE INTEGRAL AS A LIMIT OF SUM

Let $f(x)$ be a continuous real valued function defined on the closed interval [a, b] which is divided into n parts as shown in figure.



The point of division on x-axis are

$a, a+h, a+2h.........a+(n-1)h, a+nh$, where $\frac{b-a}{n} = h$.

Let S_n denotes the area of these n rectangles.

Then, $S_n = hf(a) + hf(a+h) + hf(a+2h) + ... + hf(a+(n-1)h)$

Clearly, S_n is area very close to the area of the region bounded by curve $y = f(x)$, x –axis and the ordinates x = a, x = b.

Hence $\int_a^b f(x)\,dx = \underset{n\to\infty}{\text{Lt}}\, S_n$

$$\int_a^b f(x)\,dx = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} h f(a+rh)$$

$$= \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} \left(\frac{b-a}{n}\right) f\left(a + \frac{(b-a)r}{n}\right)$$

Note...

1. We can also write

$S_n = hf(a+h) + hf(a+2h) + ... + hf(a+nh)$ and

$$\int_a^b f(x)\,dx = \underset{n\to\infty}{Lt} \sum_{r=1}^{n} \left(\frac{b-a}{n}\right) f\left(a + \left(\frac{b-a}{n}\right) r\right)$$

2. If $a = 0, b = 1$, $\int_0^1 f(x)\,dx = \underset{n\to\infty}{Lt} \sum_{r=0}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right)$

Steps to express the limit of sum as definite integral

Step 1. Replace $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx and $\underset{n\to\infty}{Lt}\ \Sigma$ by $\int$

Step 2. Evaluate $\underset{n\to\infty}{Lt} \left(\frac{r}{n}\right)$ by putting least and greatest values of r as lower and upper limits respectively.

For example $\underset{n\to\infty}{Lt} \sum_{r=1}^{pn} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^p f(x)\,dx$

$$\left[\underset{n\to\infty}{Lt} \left(\frac{r}{n}\right)\Bigg|_{r=1} = 0,\ \underset{n\to\infty}{Lt} \left(\frac{r}{n}\right)\Bigg|_{r=np} = p \right]$$

5. REDUCTION FORMULAE IN DEFINITE INTEGRALS

5.1 If $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$, then show that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

Proof: $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$

$$I_n = \left[-\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1) \sin^{n-2} x \,.\, \cos^2 x\,dx$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \,.\, (1 - \sin^2 x)\,dx$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x\,dx - (n-1) \int_0^{\frac{\pi}{2}} \sin^n x\,dx$$

$$I_n + (n-1)\, I_n = (n-1)\, I_{n-2}$$

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

Note...

1. $\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$

2. $I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right) ..... I_0$ or I_1

according as n is even or odd. $I_0 = \frac{\pi}{2}, I_1 = 1$

$$\text{Hence } I_n = \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right) ...... \left(\frac{1}{2}\right) . \frac{\pi}{2} & \text{if } n \text{ is even} \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right) .... \left(\frac{2}{3}\right) . 1 & \text{if } n \text{ is odd} \end{cases}$$

5.2 If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$, then show that $I_n + I_{n-2} = \frac{1}{n-1}$

Sol. $I_n = \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} \,.\, \tan^2 x\,dx$

$$= \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} (\sec^2 x - 1)\, dx$$

$$= \int_0^{\frac{\pi}{4}} (\tan x)^{n-2} \sec^2 x\, dx - \int_0^{\frac{\pi}{4}} (\tan x)^{n-2}\, dx$$

$$= \left[\frac{(\tan x)^{n-1}}{n-1}\right]_0^{\frac{\pi}{4}} - I_{n-2}$$

$$I_n = \frac{1}{n-1} - I_{n-2}$$

$$I_n + I_{n-2} = \frac{1}{n-1}$$

5.3 If $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x\,.\, \cos^n x\, dx$, then show that

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2}, n$$

Sol. $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^{m-1} x\, (\sin x \cos^n x)\, dx$

$$= \left[-\frac{\sin^{m-1} x . \cos^{n+1} x}{n+1}\right]_0^{\frac{\pi}{2}} +$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos^{n+1} x}{n+1}\, (m-1) \sin^{m-2} x\, \cos x\, dx$$

$$= \left(\frac{m-1}{n+1}\right) \int_0^{\frac{\pi}{2}} \sin^{m-2} x\,.\cos^n x\,.\cos^2 x\, dx$$

$$= \left(\frac{m-1}{n+1}\right) \int_0^{\frac{\pi}{2}} (\sin^{m-2} x\,.\cos^n x - \sin^m x.\cos^n x)\, dx$$

$$= \left(\frac{m-1}{n+1}\right) I_{m-2,n} - \left(\frac{m-1}{n+1}\right) I_{m,n}$$

$$\Rightarrow \left(1 + \frac{m-1}{n+1}\right) I_{m,n} = \left(\frac{m-1}{n+1}\right) I_{m-2,n}$$

$$I_{m,n} = \left(\frac{m-1}{m+n}\right) I_{m-2,n}$$

Note...

1. $I_{m,n} = \left(\frac{m-1}{m+n}\right)\left(\frac{m-3}{m+n-2}\right)\left(\frac{m-5}{m+n-4}\right) .... I_{0,n} \text{ or } I_{1,n}$

according as m is even or odd.

$$I_{0,n} = \int_0^{\frac{\pi}{2}} \cos^n x\, dx \text{ and } I_{1,n} = \int_0^{\frac{\pi}{2}} \sin x\,.\cos^n x\, dx = \frac{1}{n+1}$$

2. Walli's Formula

$$I_{m,n} = \begin{cases} \dfrac{(m-1)(m-3)(m-5)......(n-1)(n-3)(n-5).....}{(m+n)(m+n-2)(m+n-4).........}\dfrac{\pi}{2} & \text{when both m, n are even} \\[2ex] \dfrac{(m-1)(m-3)(m-5).......(n-1)(n-3)(n-5).........}{(m+n)(m+n-2)(m+n-4)........} & \text{otherwise} \end{cases}$$

AREA UNDER THE CURVES

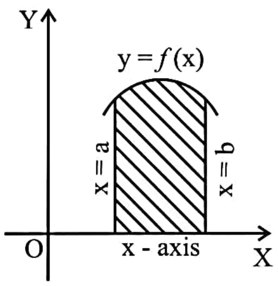
6. AREA OF PLANE REGIONS

1. The area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$. and $x = b$ (where $b > a$) is given by

$$A = \int_a^b |y|\,dx = \int_a^b |f(x)|\,dx$$

(i) If $f(x) > 0 \ \forall x \in [a, b]$

Then $A = \int_a^b f(x)\,dx$

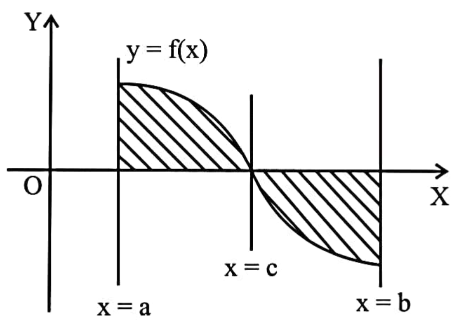


(ii) If $f(x) > 0 \ \forall x \in [a, c)$ &

$< 0 \ \forall x \in (c, b]$ Then

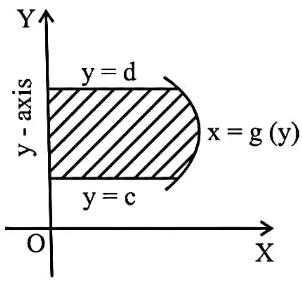
$$A = \left|\int_a^c y\,dx\right| + \left|\int_c^b y\,dx\right| = \int_a^c f(x)\,dx - \int_c^b f(x)\,dx$$

where c is a point in between a and b.



2. The area bounded by the curve $x = g(y)$, y – axis and the abscissae $y = c$ and $y = d$ (where $d > c$) is given by

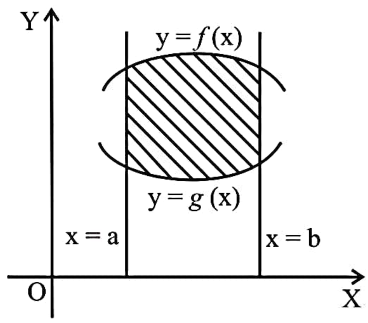
$$A = \int_c^d |x|\,dy = \int_c^d |g(y)|\,dy$$



3. If we have two curve $y = f(x)$ and $y = g(x)$, such that $y = f(x)$ lies above the curve $y = g(x)$ then the area bounded between them and the ordinates $x = a$ and $x = b$ $(b > a)$, is given by

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

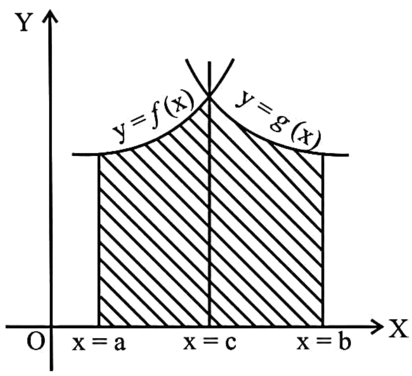
i.e. upper curve area – lower curve area.



4. The area bounded by the curves $y = f(x)$ and $y = g(x)$ between the ordinates $x = a$ and $x = b$ is given by

$$A = \int_a^c f(x)\,dx + \int_c^b g(x)\,dx,$$

where $x = c$ is the point of intersection of the two curves.



5. CURVE TRACING

In order to find the area bounded by several curves, it is important to have rough sketch of the required portion. The following steps are very useful in tracing a cartesian curve $f(x, y) = 0$.

Step 1 : Symmetry

(i) The curve is symmetrical about x-axis if all powers of y in the equation of the given curve are even.

(ii) The curve is symmetrical about y-axis if all powers of x in the equation of the given curve are even.

(iii) The curve is symmetrical about the line y = x, if the equation of the given curve remains unchanged on interchanging x and y.

(iv) The curve is symmetrical in opposite quadrants, if the equation of the given curve remains unchanged when x and y are replaced by – x and – y respectively.

Step 2 : Origin

If there is no constant term in the equation of the algebraic curve, then the curve passes through the origin.

In that case, the tangents at the origin are given by equating to zero the lowest degree terms in the equation of the given algebraic curve.

For example, the curve $y^3 = x^3 + axy$ passes through the origin and the tangents at the origin are given by axy = 0 i.e. x = 0 and y = 0.

Step 3 : Intersection with the Co-ordinate Axes

(i) To find the points of intersection of the curve with X-axis, put y = 0 in the equation of the given curve and get the corresponding values of x.

(ii) To find the points of intersection of the curve with Y-axis, put x = 0 in the equation of the given curve and get the corresponding values of y.

Step 4 : Asymptotes

Find out the asymptotes of the curve.

(i) The vertical asymptotes or the asymptotes parallel to y-axis of the given algebraic curve are obtained by equating to zero the coefficient of the highest power of y in the equation of the given curve.

(ii) The horizontal asymptotes or the asymptotes parallel to x-axis of the given algebraic curve are obtained by equating to zero the coefficient of the highest power of x in the equation of the given curve.

Step 5 : Region

Find out the regions of the plane in which no part of the curve lies. To determine such regions we solve the given equation for y in terms of x or vice-versa. Suppose that y becomes imaginary for x > a, the curve does not lie in the region x > a.

Step 6: Critical Points

Find out the values of x at which $\frac{dy}{dx} = 0$.

At such points y generally changes its character from an increasing function of x to a decreasing function of x or vice-versa.

Step 7: Trace the curve with the help of the above points.

SOLVED EXAMPLES

DEFINITE INTEGRATION

Example – 1

Evaluate the following integrals :

(i) $\int_{2}^{3} x^2\,dx$ (ii) $\int_{1}^{3} \frac{x}{(x+1)(x+2)}\,dx$

Sol. (i) $\int_2^3 x^2 dx$

$= \left[\frac{x^3}{3}\right]_2^3$

$= \frac{27}{3} - \frac{8}{3}$

$= \frac{19}{3}$

(ii) $\frac{x}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$

[Partial Fractions]

$\int_1^3 \frac{x}{(x+1)(x+2)}\,dx$

$= [-\log|x+1| + 2\log|x+2|]_1^3$

$= [-\log|4| + 2\log|5|] - [-\log|2| + 2\log|3|]$

$= [-\log 4 + 2\log 5] - [-\log 2 + 2\log 3]$

$= -2\log 2 + 2\log 5 + \log 2 - 2\log 3$

$= -\log 2 + \log 25 - \log 9 = \log 25 - \log 18$

$= \log\frac{25}{18}$

Example – 2

Evaluate : $\int_0^{\pi/4} \sec x \,.\, \sqrt{\frac{1-\sin x}{1+\sin x}}\,dx$.

Sol. $I = \int_0^{\pi/4} \sec \,.\, \sqrt{\frac{1-\sin x}{1+\sin x}}\,dx$

$= \int_0^{\pi/4} \sec x . \sqrt{\frac{1-\sin x}{1+\sin x}} \,.\, \sqrt{\frac{1-\sin x}{1-\sin x}}\,dx$

$= \int_0^{\pi/4} \sec x \frac{1-\sin x}{\sqrt{1-\sin^2 x}}\,dx$

$= \int_0^{\pi/4} \sec x \frac{1-\sin x}{\cos x}\,dx$

$= \int_0^{\pi/4} (\sec^2 x - \sec x \tan x)\,dx$

$= \int_0^{\pi/4} \sec^2 x\,dx - \int_0^{\pi/4} \sec x \tan x\,dx$

$= [\tan x]_0^{\pi/4} - [\sec x]_0^{\pi/4}$

$= \left(\tan\frac{\pi}{4} - \tan 0\right) - \left(\sec\frac{\pi}{4} - \sec 0\right)$

$= (1-0) - (\sqrt{2}-1) = 2-\sqrt{2}$.

Example – 3

Evaluate : $\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx$.

Sol. Let $I = \int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx$

Put $x^5 = t$ so that $5x^4 dx = dt$.

When $x = -1, t = -1$. When $x = 1, t = 1$.

$$I = \int_{-1}^{1} \sqrt{t+1}\,dt$$

$$= \left[\frac{(t+1)^{3/2}}{3/2}\right]_{-1}^{1} = \frac{2}{3}\left[(t+1)^{3/2}\right]_{-1}^{1}$$

$$= \frac{2}{3}[2^{3/2} - 0] = \frac{4\sqrt{2}}{3}.$$

Example – 4

Prove that $\int_{0}^{\pi/2} \sqrt{\sin\phi}\,\cos^5\phi\,d\phi = \frac{64}{231}$.

Sol. $I = \int_{0}^{\pi/2} \sqrt{\sin\phi}\,\cos^5\phi\,d\phi$

$$= \int_{0}^{\pi/2} \sqrt{\sin\phi}\,\cos^4\phi\cos\phi\,d\phi$$

$$= \int_{0}^{\pi/2} \sqrt{\sin\phi}\,(1-\sin^2\phi)^2\cos\phi\,d\phi$$

Put $\sin\phi = t$ so that $\cos\phi\,d\phi = dt$.

When $\phi = 0$, $\sin 0 = t \Rightarrow t = 0$.

When $\phi = \frac{\pi}{2}$, $\sin\frac{\pi}{2} = t \Rightarrow t = 1$

$$I = \int_{0}^{1} \sqrt{t}\,(1-t^2)^2\,dt = \int_{0}^{1} \sqrt{t}\,(\;-2t^2 + t^4)\,dt$$

$$= \int_{0}^{1} (t^{1/2} - 2t^{5/2} + t^{9/2})\,dt$$

$$= \left[\frac{t^{3/2}}{3/2} - 2\frac{t^{7/2}}{7/2} + \frac{t^{11/2}}{11/2}\right]_0^1$$

$$= \left[\frac{2}{3}t^{3/2} - \frac{4}{7}t^{7/2} + \frac{2}{11}t^{11/2}\right]_0^1$$

$$= \left[\frac{2}{3}(1) - \frac{4}{7}(1) + \frac{2}{11}(1)\right] - [0 - 0 + 0]$$

$$= \frac{2}{3} - \frac{4}{7} + \frac{2}{11}$$

$$= \frac{154 - 132 + 42}{231} = \frac{64}{231}.$$

Example – 5

Evaluate : $\int_{1}^{2} \left(\frac{x-1}{x^2}\right) e^x dx$

Or

$\int_{1}^{2} e^x\left(\frac{1}{x} - \frac{1}{x^2}\right) dx$

Sol. $\int \left(\frac{x-1}{x^2}\right) e^x dx = \int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right) dx$

$$= \int \frac{1}{x}.e^x dx - \int \frac{1}{x^2}.e^x dx$$

$$= \frac{1}{x}.e^x - \int\left(-\frac{1}{x^2}\right)e^x dx - \int \frac{1}{x^2}.e^x dx$$

[Integrating first integral by parts]

$$= \frac{1}{x}.e^x = F(x)$$

$$\int_{1}^{2} \left(\frac{x-1}{x^2}\right) e^x dx = \left[\frac{e^x}{x}\right]_1^2$$

$$= \frac{1}{2}.e^2 - \frac{1}{1}e^1 = \frac{1}{2}e^2 - e.$$

AREA UNDER THE CURVES

Example - 1

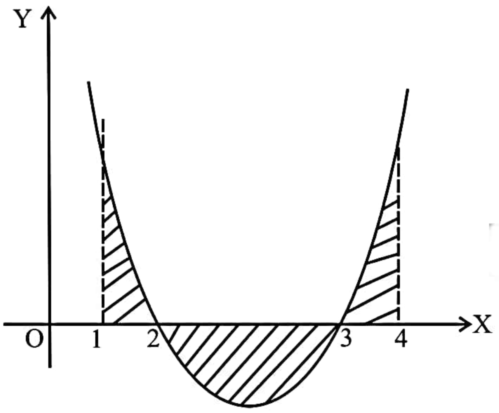
Find the area bounded by the curve $y = x^2 - 5x + 6$, X–axis and the lines $x = 1$ and 4.

Sol. For $y = 0$, we get $x^2 + 5x + 6 = 0$

$\Rightarrow\ x = 2, 3$

Hence the curve crosses X–axis at $x = 2, 3$ in the interval [1, 4].

$$\text{Bounded Area} = \left|\int_1^2 y\,dx\right| + \left|\int_2^3 y\,dx\right| + \left|\int_3^4 y\,dx\right|$$



$$\Rightarrow\ A = \left|\int_1^2 (x^2 - 5x + 6)\,dx\right| + \left|\int_2^3 (x^2 - 5x + 6)\,dx\right| + \left|\int_3^4 (x^2 - 5x + 6)\,dx\right|$$

$$A_1 = \left[\frac{2^3 - 1^3}{3}\right] - 5\left(\frac{2^2 - 1^2}{2}\right) + [6(2-1)] = \frac{5}{6}$$

$$A_2 = \frac{3^3 - 2^3}{3} - 5\left(\frac{3^2 - 2^2}{2}\right) + 6(3-2) = -\frac{1}{6}$$

$$A_3 = \frac{4^3 - 3^3}{3} - 5\left(\frac{4^2 - 3^2}{2}\right) + 6(4-3) = \frac{5}{6}$$

$$\Rightarrow\ A = \frac{5}{6} + \left|-\frac{1}{6}\right| + \frac{5}{6} = \frac{11}{6}\ \text{sq. units.}$$

Example - 2

Find the area bounded by the curve : $y = \sqrt{4 - x}$, X–axis and Y–axis.

Sol. Trace the curve $y = \sqrt{4 - x}$.

1. Put $y = 0$ in the given curve to get $x = 4$ as the point of intersection with X–axis.

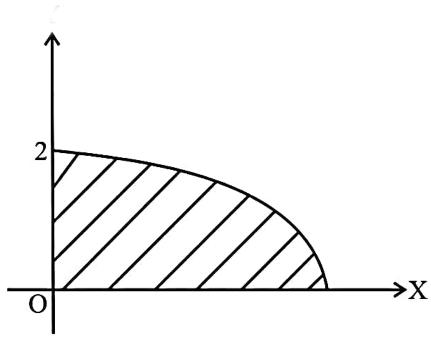
   Put $x = 0$ in the given curve to get $y = 2$ as the point of intersection with Y–axis.

2. For the curve, $y = \sqrt{4 - x}$, $4 - x \geq 0$

   $\Rightarrow\ x \leq 4$

   $\Rightarrow$ curve lies only to the left of $x = 4$ line.

3. As any y is positive, curve is above X–axis.



Using step 1 to 3, we can draw the rough sketch of $y = \sqrt{4 - x}$.

In figure,

$$\text{Bounded area} = \int_0^4 \sqrt{4 - x}\,dx = \left|\frac{-2}{3}(4 - x)\sqrt{4 - x}\right|_0^4$$

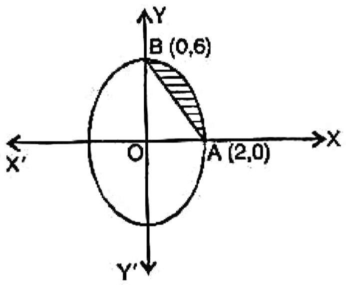
$$= \frac{16}{3}\ \text{sq. units.}$$

Example - 3

AOBA is the part of the ellipse $9x^2 + y^2 = 36$ in the first quadrant such that OA = 2 and OB = 6. Find the area between the arc AB and the chord AB.

Sol. The given equation of the ellipse can be written as

$$\frac{x^2}{4}+\frac{y^2}{36}=1 \text{ i.e. } \frac{x^2}{2^2}+\frac{y^2}{6^2}=1$$



A is (2, 0) and B is (0, 6).

The equation of chord AB is :

$$y-0=\frac{6-0}{0-2}(x-2)$$

$\Rightarrow$ $y = -3x + 6$.

Reqd. area (shown shaded)

$$=\int_0^2 3\sqrt{4-x^2}\,dx-\int_0^2(6-3x)\,dx$$

$$=3\left[\frac{x\sqrt{4-x^2}}{2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2-\left[6x-\frac{3x^2}{2}\right]_0^2$$

$$=3\left[\frac{2}{2}(0)+2\sin^{-1}(1)\right]-\left[6(2)-\frac{3(4)}{2}\right]$$

$$=3\left[2\times\frac{\pi}{2}\right]-[12-6]$$

$= (3\pi - 6)$ sq. units.

Example - 4

Find the area bounded by the curves $y = x^2$ and $x^2 + y^2 = 2$ above X–axis.

Sol. Let us first find the points of intersection of curves.

Solving $y = x^2$ and $x^2 + y^2 = 2$ simultaneously, we get :

$x^2 + x^4 = 2$

$\Rightarrow$ $(x^2 - 1)(x^2 + 2) = 0$

$\Rightarrow$ $x^2 = 1$ and $x^2 = -2$ [reject]

$\Rightarrow$ $x = \pm 1$

$\Rightarrow$ $A = (-1, 1)$ and $B = (1, 1)$

$$\text{Shaded Area} = \int_{-1}^{+1}\left(\sqrt{2-x^2}-x^2\right)dx$$

$$=\int_{-1}^{+1}\sqrt{2-x^2}\,dx-\int_{-1}^{+1}x^2\,dx$$

$$=2\int_0^1\sqrt{2-x^2}\,dx-2\int_0^1 x^2\,dx$$

$$=2\left[\frac{x}{2}\sqrt{2-x^2}+\frac{2}{2}\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^1-2\left(\frac{1}{3}\right)$$

$$=2\left(\frac{1}{2}+\frac{\pi}{4}\right)-\frac{2}{3}=\frac{1}{3}+\frac{\pi}{2}\text{ sq. units.}$$