# Chapter 4 - Determinants

- System of algebraic equations can be expressed in the form of matrices.
  - Linear Equations Format

    $a_1x + b_1y = c_1$

    $a_2x + b_2y = c_2$
  - Matrix Format:

    $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$
- The values of the variables satisfying all the linear equations in the system, is called solution of system of linear equations.
- If the system of linear equations has a unique solution. This unique solution is called determinant of Solution or det A

## Applications of Determinants

- Engineering
- Science
- Economics
- Social Science, etc.

## Determinant

- A determinant is defined as a (mapping) unction from the set o square matrices to the set of real numbers
- Every square matrix A is associated with a number, called its determinant
- Denoted by det (A) or |A| or Δ
- If A $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(A)$
- Only square matrices have determinants.
- The matrices which are not square do not have determinants
- For matrix A, |A| is read as determinant of A and not modulus of A.

## Types of Determinant

1. **First Order Determinant**
   - Let A = [a ] be the matrix of order 1, then determinant of A is defined to be equal to a
   - If A = [a], then det (A) = |A| = a
2. **Second Order Determinant**
   - Let A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ matrix of order 2x2

     $$\det(A) = |A| = \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$
   - Eg. Evaluate $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$

     $= 2(2) - 4(-1) = 4 + 4 = 8$

## 3. Third Order Determinant

- Can be determined by expressing it in terms of second order determinants
- Let $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

The below method is explained for expansion around Row 1

The value of the determinant, thus will be the sum of the product of element in line parallel to the diagonal minus the sum of the product of elements in line perpendicular to the line segment. Thus,

Then det A = |A| = $a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$$|A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

The same procedure can be repeated for Row 2, Row 3, Column 1, Column 2, and Column 3

- Note
  - Expanding a determinant along any row or column gives same value.
  - This method doesn't work for determinants of order greater than 3.
  - For easier calculations, we shall expand the determinant along that row or column which contains maximum number of zeros
  - In general, if A = kB where A and B are square matrices of order n, then $|A| = k^n |B|$, where n = 1, 2, 3

## Properties of Determinants

- Helps in simplifying its evaluation by obtaining maximum number of zeros in a row or a column.
- These properties are true for determinants of any order.

### Property 1

- The value of the determinant remains unchanged if its rows and columns are interchanged
- **Verification:**

Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Expanding along first row, we get

$$\Delta = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

By interchanging the rows and columns of Δ, we get the determinant

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding Δ1 along first column, we get

$$\Delta_1 = a_1 (b_2 c_3 - c_2 b_3) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)$$

Hence $\Delta = \Delta_1$

Note:

- It follows from above property that if A is a square matrix,
  Then det (A) = det (A′), where A′ = transpose of A
- If Ri = ith row and Ci = ith column, then for interchange of row and
- columns, we will symbolically write Ci ↔ Ri

**Property 2**

- If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.
- **Verification:**

Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Expanding along first row, we get

$$\Delta = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

By interchanging the first and third rows of Δ, we get the determinant

$$\Delta_1 = \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Expanding along third row, we get

$$\Delta_1 = a_1(c_2b_3 - b_2c_3) - a_2(c_1b_3 - c_3b_1) + a_3(b_2c_1 - b_1c_2)$$

$$= -\left[a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)\right]$$

Hence $\Delta_1 = -\Delta$

**Property 3**

- If any two rows (or columns) of a determinant are identical (all corresponding elements are same), then value of determinant is zero.
- **Verification:**
- Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ in which two rows are identical i.e. $a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$,
  - If we interchange the identical rows (or columns) of the determinant Δ, then Δ does not change.
  - However, by Property 2, it follows that Δ has changed its sign
  - Therefore $\Delta = -\Delta$ or $\Delta = 0$

- **Property 4**
  - If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value gets multiplied by k
  - **Verification**

    Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

    Let $\Delta_1$ be the determinant obtained by multiplying the elements of the first row by k.

    Then $\Delta_1 = \begin{vmatrix} k a_1 & k b_1 & k c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

    Expanding along first row, we get

    $\Delta_1 = k\, a_1(b_2\, c_3 - b_3\, c_2) - k\, b_1(a_2\, c_3 - c_2\, a_3) + k\, c_1(a_2\, b_3 - b_2\, a_3)$

    $= k\,[a_1(b_2\, c_3 - b_3\, c_2) - b_1(a_2\, c_3 - c_2\, a_3) + c_1(a_2\, b_3 - b_2\, a_3)]$

    $= k\Delta$

    Hence $\begin{vmatrix} k a_1 & k b_1 & k c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

- **Property 5**
  - If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.
  - **Verification:**

    Take $\begin{vmatrix} a_1+\lambda_1 & a_2+\lambda_2 & a_3+\lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

    LHS = $\begin{vmatrix} a_1+\lambda_1 & a_2+\lambda_2 & a_3+\lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

    Expanding the determinants along the first row, we get

    $\Delta = (a_1+\lambda_1)(b_2c_3 - c_2b_3) - (a_2+\lambda_2)(b_1c_3 - b_3c_1) + (a_3+\lambda_3)(b_1c_2 - b_2c_1)$

    $= a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) + \lambda_1(b_2c_3 - c_2b_3) - \lambda_2(b_1c_3 - b_3c_1) + \lambda_3(b_1c_2 - b_2c_1)$

    (by rearranging terms)

    $= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = R.H.S.$

- **Property 6**
- If, to each element of any row or column of a determinant, the equimultiples of corresponding elements of other row (or column) are added, then value of determinant remains the same, i.e., the value of determinant remain same if we apply the operation

$$R_i \to R_i + kR_j \text{ or } C_i \to C_i + kC_j.$$

- **Verification**

  Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} a_1 + kc_1 & a_2 + kc_2 & a_3 + kc_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

  Where $\Delta_1$ is obtained by the operation $R_1 \to R_1 + kR_3$ .
  Here, we have multiplied the elements of the third row (R3) by a constant k and added them to the corresponding elements of the first row (R1).

  Symbolically, we write this operation as $R_1 \to R_1 + k R_3$

  $$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} kc_1 & kc_2 & kc_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{ (Using Property 5)}$$

  $= \Delta + 0$ (since R1 and R3 are proportional)

  Hence $\Delta = \Delta_1$

- **Property 7**
  - If each element of a row (or column) of a determinant is zero, then its value is zero

- **Property 8**
  - In a determinant, If all the elements on one side of the principal diagonal are Zero's , then the value of the determinant is equal to the product of the elements in the principal diagonal

## Area of a Triangle

- Let $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ be the vertices of a triangle, then

Area of Triangle = $\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

- Note
  - Area is a positive quantity, we always take the absolute value of the determinant
  - If area is given, use both positive and negative values of the determinant for calculation.
  - The area of the triangle formed by three collinear points is zero.

## Minors and Cofactors

### Minor

- If the row and column containing the element $a_{11}$ (i.e., $1^{st}$ row and $1^{st}$ column)are removed, we get the second order determinant which is called the Minor of element $a_{11}$
- Minor of an element $a_{ij}$ of a determinant is the determinant obtained by deleting its ith row and jth column in which element $a_{ij}$ lies.

- Minor of an element $a_{ij}$ is denoted by $M_{ij}$
- Minor of an element of a determinant of order $n(n \geq 2)$ is a determinant of order $n - 1$
- Eg: Find Minor o the element 6 in the determinant A given
  - Let $\Delta A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

    Since 6 lies in the second row and third column, its minor $M_{23}$ is given by

    $M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 8 - 14 = -6$ (Obtained by deleting R2 and C3 in Δ)

**Cofactor**

- If the minors are multiplied by the proper signs we get cofactors
- The cofactor of the element $a_{ij}$ is $C_{ij} = (-1)^{i+j} M_{ij}$
- The signs to be multiplied are given by the rule

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

- If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero.

$$\Delta = a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23}$$

$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \text{ (since } R_1 \text{ and } R_2 \text{ are identical)}$$

  - Eg: Find the cofactors of the element 4 in the given determinant A
    - Let $A = \begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$

Cofactor of 4 is $A_{12} = (-1)^{1+2} \; M_{12} = (-1)^3(4) = -4$

## Adjoint and Inverse of a Matrix

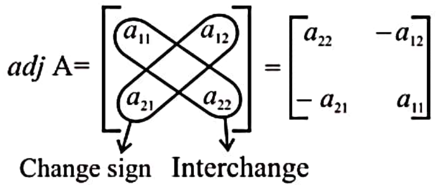
- Adjoint of a matrix is the transpose of the matrix of cofactors of the given matrix
- Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
- Matrix formed by the cofactors are $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$$adj\ A = \text{Transpose of} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

- Note
  - For a square matrix of order 2, given by

    $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

    The adj A can also be obtained by interchanging $a_{11}$ and $a_{22}$ and by changing signs Of $a_{12}$ and $a_{21}$, i.e.




$$adj\ A = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

## Theorem 1

- If A be any given square matrix of order n,
  Then A (adj A) = (adj A) A = A I,
  Where I is the identity matrix of order n

- **Verification:**

  Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then adj A= $\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

  Since sum of product of elements of a row (or a column) with corresponding cofactors is equal to |A| and otherwise zero, we have

  $$A\,(adj\ A) = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\, I$$

  Similarly, we can show (adj A) A = A I
  Hence A (adj A) = (adj A) A = A I

## Singular & No Singular Matrix:

- A square matrix A is said to be singular if |A| = 0
- A square matrix A is said to be non-singular if |A | ≠ 0

## Theorem 2

  - If A and B are non-singular matrices of the same order, then AB and BA are also non-singular matrices of the same order.

## Theorem 3

- The determinant of the product of matrices is equal to product of their respective determinants, that is, AB =|A| |B| , where A and B are square matrices of the same order

**Verification**

We know that, $(adj\ A)A = |A|I = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$

Writing determinants of matrices on both sides, we have

$$|(adj\ A)A| = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$|(adj\ A)||A| = |A^3| \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$|(adj\ A)||A| = |A^3|(1)$$

$$|(adj\ A)| = |A|^2$$

In general, if A is a square matrix of order n, then $|\text{adj}(A)| = |A|^{n-1}$

## Theorem 4

- A square matrix A is invertible if and only if A is non-singular matrix.

**Verification**

Let A be invertible matrix of order n and I be the identity matrix of order n. Then, there exists a square matrix B of order n such that AB = BA = I

Now AB = I. So |AB| = I or |A| |B| = 1 (since |I|= 1, |AB|=| A||B|). This gives |A| ≠ 0. Hence A is non-singular.

Conversely, let A be non-singular. Then |A| ≠ 0

Now A (adj A) = (adj A) A = |A| I (Theorem 1)

$$A\left(\frac{1}{|A|}adj\,A\right) = \left(\frac{1}{|A|}adj\ A\right)A = I$$

$$AB = BA = I, \text{ where } B = \frac{1}{|A|}adj\ A$$

$$A \text{ is invertible and } A^{-1} = \frac{1}{|A|}adj\ A$$

## Applications of Determinants and Matrices

- Used for solving the system of linear equations in two or three variables and for checking the consistency of the system of linear equations.
- **Consistent system**
  - A system of equations is said to be consistent if its solution (one or more) exists.
- **Inconsistent system**
  - A system of equations is said to be inconsistent if its solution does not exist

## Solution of system of linear equations using inverse of a matrix

Let the system of Equations be as below:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then, the system of equations can be written as, AX = B, i.e.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

### Case 1:

If A is a non-singular matrix, then its inverse exists.

$$AX = B$$
$$A^{-1}(AX) = A^{-1}B \quad \text{(premultiplying by } A^{-1}\text{)}$$
$$(A^{-1}A)X = A^{-1}B \quad \text{(by associative property)}$$
$$1X = A^{-1}B$$
$$X = A^{-1}B$$

This matrix equation provides unique solution for the given system of equations as inverse of a matrix is unique. This method of solving system of equations is known as **Matrix Method**

### Case II

If A is a singular matrix, then |A| = 0.

In this case, we calculate (adj A) B.

If (adj A) B ≠ O, (O being zero matrix), then solution does not exist and the system of equations is called inconsistent.

If (adj A) B = O, then system may be either consistent or inconsistent according as the system have either infinitely many solutions or no solution

### Summary

For a square matrix A in matrix equation AX = B

- |A| ≠ 0, there exists unique solution
- |A| = 0 and (adj A) B ≠ 0, then there exists no solution
- |A| = 0 and (adj A) B = 0, then system may or may not be consistent.