

Chapter 1 - Relations and Functions

Definitions:

Let A and B be two non-empty sets, then a function f from set A to set B is a rule which associates each element of A to a unique element of B.

○ **Relation**

If $(a, b) \in R$, we say that a is related to b under the relation R and we write as $a R b$

○ **Function**

It is represented as $f: A \rightarrow B$ and function is also called mapping.

○ **Real Function**

$f: A \rightarrow B$ is called a real function, if A and B are subsets of R.

○ **Domain and Codomain of a Real Function**

Domain and codomain of a function f is a set of all real numbers x for which $f(x)$ is a real number. Here, set A is domain and set B is codomain.

○ **Range of a real function**

f is a set of values $f(x)$ which it attains on the points of its domain

Types of Relations

- A relation R in a set A is called **Empty relation**, if no element of A is related to any element of A, i.e., $R = \varnothing \subset A \times A$.
- A relation R in a set A is called **Universal relation**, if each element of A is related to every element of A, i.e., $R = A \times A$.
- Both the empty relation and the universal relation are sometimes called **Trivial Relations**
- A relation R in a set A is called
 - **Reflexive**
 - if $(a, a) \in R$, for every $a \in A$,
 - **Symmetric**
 - If $(a_1, a_2) \in R$ implies that $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.
 - **Transitive**
 - If $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ implies that $(a_1, a_3) \in R$, for all $a_1, a_2, a_3 \in A$.
- A relation R in a set A is said to be an **equivalence relation** if R is reflexive, symmetric and transitive
- The set E of all even integers and the set O of all odd integers are subsets of Z satisfying following conditions:
 - All elements of E are related to each other and all elements of O are related to each other.
 - No element of E is related to any element of O and vice-versa.
 - E and O are disjoint and $Z = E \cup O$.
 - The subset E is called the equivalence class containing zero, Denoted by [0].
 - O is the equivalence class containing 1 and is denoted by [1].

○ **Note**

- $[0] \neq [1]$
- $[0] = [2r]$
- $[1] = [2r + 1], r \in \mathbb{Z}.$

- Given an arbitrary equivalence relation R in an arbitrary set X , R divides X into mutually disjoint subsets A_i called partitions or subdivisions of X satisfying:
 - All elements of A_i are related to each other, for all i .
 - No element of A_i is related to any element of $A_j, i \neq j$.
 - $\bigcup A_j = X$ and $A_i \cap A_j = \emptyset, i \neq j$.
- The subsets A_i are called equivalence classes.

Note:

- Two ways of representing a relation
 - Roaster method
 - Set builder method
- If $(a, b) \in R$, we say that a is related to b and we denote it as **$a R b$** .

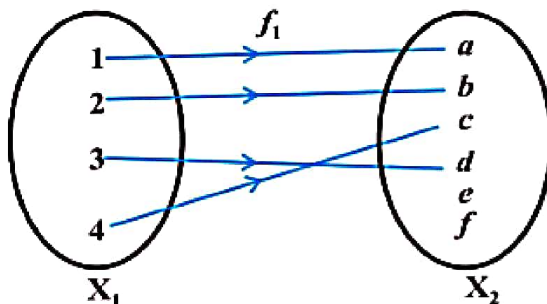
Types of Functions

Consider the functions f_1, f_2, f_3 and f_4 given

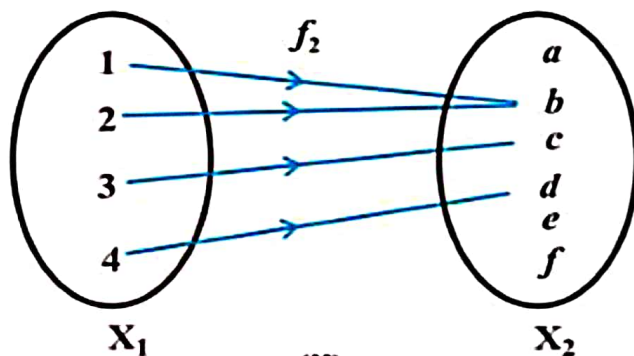
- A function $f: X \rightarrow Y$ is defined to be **one-one (or injective)**, if the images of distinct elements of X under f are distinct, i.e., for every $x_1, x_2 \in X, f(x_1) = f(x_2)$ implies $x_1 = x_2$. Otherwise, f is called **many-one**.

Example

- One- One Function

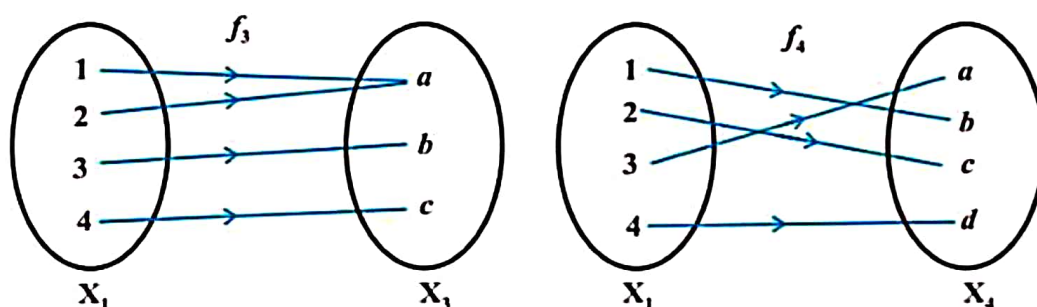


- Many-One Function

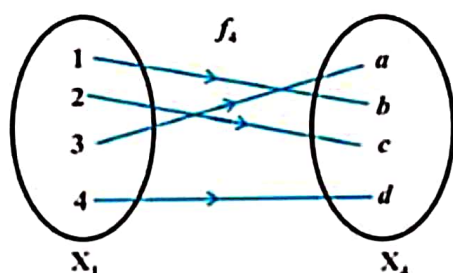


- A function $f: X \rightarrow Y$ is said to be **onto (or surjective)**, if every element of Y is the image of some element of X under f , i.e., for every $y \in Y$, there exists an element x in X such that $f(x) = y$.

- $f: X \rightarrow Y$ is onto if and only if $\text{Range of } f = Y$.
- Eg:



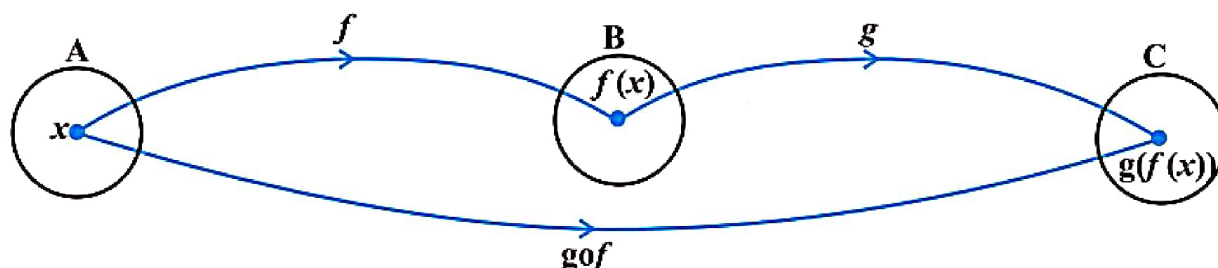
- A function $f: X \rightarrow Y$ is said to be **one-one and onto (or bijective)**, if f is both one-one and onto.
- Eg:



Composition of Functions and Invertible Function

Composite Function

- Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two functions.
- Then the composition of f and g , denoted by $g \circ f$, is defined as the function $g \circ f: A \rightarrow C$ given by $g \circ f(x) = g(f(x))$, $\forall x \in A$.

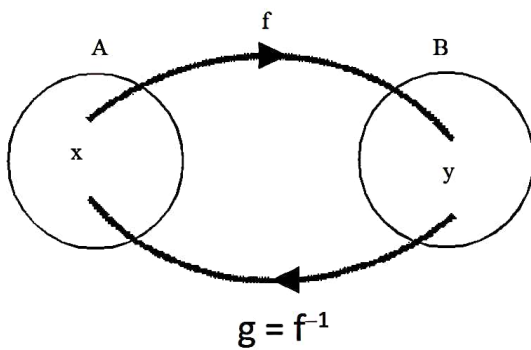


- Eg:
 - Let $f: \{2, 3, 4, 5\} \rightarrow \{3, 4, 5, 9\}$ and $g: \{3, 4, 5, 9\} \rightarrow \{7, 11, 15\}$ be functions
 - Defined as $f(2) = 3, f(3) = 4, f(4) = f(5) = 5$ and $g(3) = g(4) = 7$ and $g(5) = g(9) = 11$.
 - Find $g \circ f$.
 - Solution
 - $g \circ f(2) = g(f(2)) = g(3) = 7$,
 - $g \circ f(3) = g(f(3)) = g(4) = 7$,
 - $g \circ f(4) = g(f(4)) = g(5) = 11$ and
 - $g \circ f(5) = g(5) = 11$
- It can be verified in general that $g \circ f$ is one-one implies that f is one-one. Similarly, $g \circ f$ is onto implies that g is onto.

- While composing f and g , to get $g \circ f$, first f and then g was applied, while in the reverse process of the composite $g \circ f$, first the reverse process of g is applied and then the reverse process of f .
- If $f: X \rightarrow Y$ is a function such that there exists a function $g: Y \rightarrow X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$, then f must be one-one and onto.

Invertible Function

- A function $f: X \rightarrow Y$ is defined to be invertible, if there exists a function $g: Y \rightarrow X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$. The function g is called the inverse of f .
- Denoted by f^{-1} .



- Thus, if f is invertible, then f must be one-one and onto and conversely, if f is one-one and onto, then f must be invertible.

Theorem 1

- If $f: X \rightarrow Y$, $g: Y \rightarrow Z$ and $h: Z \rightarrow S$ are functions, then
 - $h \circ (g \circ f) = (h \circ g) \circ f$.
- Proof
We have
 - $h \circ (g \circ f)(x) = h(g \circ f(x)) = h(g(f(x))), \forall x \text{ in } X$
 - $(h \circ g) \circ f(x) = h \circ g(f(x)) = h(g(f(x))), \forall x \text{ in } X$.

Hence, $h \circ (g \circ f) = (h \circ g) \circ f$

Theorem 2

- Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be two invertible functions.
 - Then $g \circ f$ is also invertible with $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$
- Proof
 - To show that $g \circ f$ is invertible with $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, it is enough to show that $(f^{-1} \circ g^{-1}) \circ (g \circ f) = I_X$ and $(g \circ f) \circ (f^{-1} \circ g^{-1}) = I_Z$.
Now, $(f^{-1} \circ g^{-1}) \circ (g \circ f) = ((f^{-1} \circ g^{-1}) \circ g) \circ f$, by Theorem 1
 $= (f^{-1} \circ (g^{-1} \circ g)) \circ f$, by Theorem 1
 $= (f^{-1} \circ I_Y) \circ f$, by definition of g^{-1}
 $= I_X$

Similarly, it can be shown that $(g \circ f) \circ (f^{-1} \circ g^{-1}) = I_Z$

Binary Operations

Definitions:

- A binary operation $*$ on a set A is a function $*$: $A \times A \rightarrow A$. We denote $*(a, b)$ by $a * b$.

- A binary operation $*$ on the set X is called commutative, if $a * b = b * a$, for every $a, b \in X$
- A binary operation $* : A \times A \rightarrow A$ is said to be associative if $(a * b) * c = a * (b * c)$, $\forall a, b, c, \in A$.
- A binary operation $* : A \times A \rightarrow A$, an element $e \in A$, if it exists, is called identity for the operation $*$, if $a * e = a = e * a$, $\forall a \in A$.
 - Zero is identity for the addition operation on \mathbb{R} but it is not identity for the addition operation on \mathbb{N} , as $0 \notin \mathbb{N}$.
 - Addition operation on \mathbb{N} does not have any identity.
 - For the addition operation $+ : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, given any $a \in \mathbb{R}$, there exists $-a$ in \mathbb{R} such that $a + (-a) = 0$ (identity for '+') $= (-a) + a$.
 - For the multiplication operation on \mathbb{R} , given any $a \neq 0$ in \mathbb{R} , we can choose $\frac{1}{a}$ such that $a \times \frac{1}{a} = 1$ (identity for '×') $= 1 = \frac{1}{a} \times a$
- A binary operation $* : A \times A \rightarrow A$ with the identity element e in A , an element $a \in A$ is said to be invertible with respect to the operation $*$, if there exists an element b in A such that $a * b = e = b * a$ and b is called the inverse of a and is denoted by a^{-1}